

SHARP MORREY-SOBOLEV INEQUALITIES ON COMPLETE RIEMANNIAN MANIFOLDS

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ABSTRACT. Two Morrey-Sobolev inequalities (with support-bound and L^1 -bound, respectively) are investigated on complete Riemannian manifolds with their sharp constants in \mathbb{R}^n . We prove the following results in both cases:

- If (M, g) is a *Cartan-Hadamard manifold* which verifies the n -dimensional Cartan-Hadamard conjecture, sharp Morrey-Sobolev inequalities hold on (M, g) . Moreover, extremals exist if and only if (M, g) is isometric to the standard Euclidean space (\mathbb{R}^n, e) .
- If (M, g) has *non-negative Ricci curvature*, (M, g) supports the sharp Morrey-Sobolev inequalities if and only if (M, g) is isometric to (\mathbb{R}^n, e) .

1. INTRODUCTION AND MAIN RESULTS

One of the most important topics of Sobolev inequalities is to find sharp constants and extremals in the embeddings $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, where the numbers $p, q \in \mathbb{R}$ and $n \in \mathbb{N}$ are related in the Sobolev sense. Owing to a systematic study initiated by T. Aubin and G. Talenti in the middle of seventies, various results are available nowadays concerning sharp constants and extremals in Sobolev inequalities both in the Euclidean and Riemannian frameworks; see Ghoussoub and Moradifard [11], Hebey [12], Maz'ya [15], and references therein. We emphasize that sharp Sobolev inequalities in \mathbb{R}^n were mostly studied for $p \in [1, n]$; see the pioneering works of Federer and Fleming [10] when $p = 1$, and Aubin [1] and Talenti [18] when $1 < p < n$. Moreover, when $p \in [1, n]$, several rigidity results can be found on Riemannian manifolds supporting Sobolev-type inequalities with their Euclidean sharp constants, see Ledoux [14], do Carmo and Xia [7], Druet and Hebey [8], Druet, Hebey and Vaigon [9], and the comprehensive monograph of Hebey [12].

The main purpose of this paper is to investigate *sharp Morrey-Sobolev inequalities* (i.e., $p > n$) on non-compact complete Riemannian manifolds having either non-positive sectional curvature or non-negative Ricci curvature. Hereafter, sharpness means that a given inequality on the Riemannian manifold is valid with its Euclidean sharp constant.

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The Morrey-Sobolev inequality in \mathbb{R}^n states that the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ is continuous when $p > n$ (see [3], [15]), i.e., there exists $C(p, n) > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(p, n)(\|u\|_{L^p(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)}), \quad \forall u \in W^{1,p}(\mathbb{R}^n).$$

A generic Morrey-Sobolev inequality has been established on smooth complete n -dimensional Riemannian manifolds with Ricci curvature bounded from below and verifying a global volume growth assumption, see Coulhon [5].

Let (M, g) be an $n(\geq 2)$ -dimensional smooth complete Riemannian manifold. In order to present our results, we need two notions. First, we say that a function $u : M \rightarrow [0, \infty)$ is *concentrated around* $x_0 \in M$ if for every $0 < t < \|u\|_{L^\infty(M)}$, the level set $\{x \in M : u(x) > t\}$ is a geodesic ball $B(x_0, \rho_t) = \{x \in M : d(x_0, x) < \rho_t\}$ for some $\rho_t > 0$. Hereafter, $d : M \times M \rightarrow \mathbb{R}$ denotes the usual distance function associated with g . Second, Morrey-Sobolev inequalities will be particularly investigated on Cartan-Hadamard manifolds as well (i.e., on simply connected, complete Riemannian manifolds with non-positive sectional curvature) where the validity of the *Cartan-Hadamard conjecture* will play an indispensable role. For the sake of completeness, we recall the

Cartan-Hadamard conjecture in n -dimension (see Aubin [1]). *Let (M, g) be an n -dimensional Cartan-Hadamard manifold. Then any compact domain $D \subset M$ with smooth boundary ∂D satisfies the Euclidean isoperimetric inequality, i.e.,*

$$(1.1) \quad \text{Area}_g(\partial D) \geq n\omega_n^{\frac{1}{n}} \text{Vol}_g(D)^{\frac{n-1}{n}}.$$

Moreover, equality holds in (1.1) if and only if D is isometric to the n -dimensional Euclidean ball with volume $\text{Vol}_g(D)$.

Hereafter, ω_n is the volume of the n -dimensional Euclidean unit ball; $\text{Area}_g(\partial D)$ stands for the area of ∂D with respect to the metric induced on ∂D by g ; and $\text{Vol}_g(D)$ is the volume of D with respect to g .

Remark 1.1. Cartan-Hadamard conjecture is true in dimension 2, see Weil [16]; in dimension 3, see Kleiner [13]; and in dimension 4, see Croke [6], but it is open for higher dimensions.

Now we are ready to present our main results.

(I) Sharp Morrey-Sobolev inequality with support-bound. Let (M, g) be an $n(\geq 2)$ -dimensional smooth complete Riemannian manifold and $p > n$. For some $C > 0$, we consider on (M, g) the Morrey-Sobolev inequality

$$\|u\|_{L^\infty(M)} \leq C \mathcal{H}^n(\text{sprt } u)^{\frac{1}{n} - \frac{1}{p}} \|\nabla_g u\|_{L^p(M)}, \quad \forall u \in \text{Lip}_0(M). \quad (\text{MS})_C^1$$

Here, $\text{sprt } u$ is the support of u , \mathcal{H}^n is the n -dimensional Hausdorff measure on M , $\|\nabla_g u\|_{L^p(M)}$ stands for the $L^p(M)$ norm of the vector $\nabla_g u(x) \in T_x M$, while $\text{Lip}_0(M)$ is the space of Lipschitz functions with compact support defined on M . Although we can put $C_0^\infty(M)$ instead of $\text{Lip}_0(M)$ in $(\text{MS})_C^1$ due to density reasons, we prefer the latter choice taking into account the specific shape of extremals in the Euclidean setting. Indeed, by using symmetrization and rearrangement arguments,

Talenti [19, Theorem 2.E] proved that if $(M, g) = (\mathbb{R}^n, e)$ is the standard Euclidean space, then $(\mathbf{MS})_{C_1(p,n)}^1$ holds with the sharp constant

$$(1.2) \quad C_1(p, n) = n^{-\frac{1}{p}} \omega_n^{-\frac{1}{n}} \left(\frac{p-1}{p-n} \right)^{\frac{1}{p'}},$$

where $p' = \frac{p}{p-1}$, and the unique class of extremals (up to a constant multiplication) is given by

$$(1.3) \quad u_{\lambda, x_0}(x) = \left(\lambda^{\frac{p-n}{p-1}} - |x - x_0|^{\frac{p-n}{p-1}} \right)_+, \quad x \in \mathbb{R}^n,$$

where $\lambda > 0$, $x_0 \in \mathbb{R}^n$, and $r_+ = \max(r, 0)$. Clearly, for a fixed $x_0 \in \mathbb{R}^n$, the function $u_{\lambda, x_0} \in \text{Lip}_0(\mathbb{R}^n)$ is concentrated around x_0 , its support being the closed Euclidean ball $B_e[x_0, \lambda]$.

Our first result reads as follows.

Theorem 1.1. *Let (M, g) be an n -dimensional Cartan-Hadamard manifold which verifies the Cartan-Hadamard conjecture in the same dimension, and let $p > n$.*

- (i) [Sharpness] *The Morrey-Sobolev inequality $(\mathbf{MS})_{C_1(p,n)}^1$ holds on (M, g) ; moreover, $C_1(p, n)$ is sharp, i.e.,*

$$C_1(p, n)^{-1} = \inf_{u \in \text{Lip}_0(M) \setminus \{0\}} \frac{\mathcal{H}^n(\text{sprt } u)^{\frac{1}{n} - \frac{1}{p}} \|\nabla_g u\|_{L^p(M)}}{\|u\|_{L^\infty(M)}}.$$

- (ii) [Extremals] *Let $x_0 \in M$. The following statements are equivalent:*

- (a) *For every $\kappa > 0$ there exists a non-negative extremal function $u \in \text{Lip}_0(M)$ in $(\mathbf{MS})_{C_1(p,n)}^1$, concentrated around x_0 and $\mathcal{H}^n(\text{sprt } u) = \kappa$;*
(b) *(M, g) is isometric to (\mathbb{R}^n, e) .*

In the non-negatively curved case we state the following result:

Theorem 1.2. *Let (M, g) be a complete, n -dimensional Riemannian manifold with non-negative Ricci curvature, let $p > n$, and assume that $(\mathbf{MS})_C^1$ holds on (M, g) for some $C > 0$. Then the following assertions hold:*

- (i) $C \geq C_1(p, n)$;
(ii) (M, g) has the large volume balls property, i.e.,

$$\text{Vol}_g(B(x, \rho)) \geq \left(\frac{C_1(p, n)}{C} \right)^{\frac{pn}{p-n}} \omega_n \rho^n \quad \text{for all } x \in M, \rho \geq 0;$$

- (iii) $(\mathbf{MS})_{C_1(p,n)}^1$ holds on (M, g) if and only if (M, g) is isometric to (\mathbb{R}^n, e) .

(II) **Sharp Morrey-Sobolev inequality with L^1 -bound.** Instead of having a support-bound estimate in term of $\mathcal{H}^n(\text{sprt } u)$ for $\|u\|_{L^\infty(M)}$, we can use a suitable interpolation between $\|\nabla_g u\|_{L^p(M)}$ and some other norm $\|u\|_{L^q(M)}$, $q \in [1, \infty)$. To do this, let (M, g) be a smooth n -dimensional complete Riemannian manifold and $p > n$. For some $C > 0$, we consider on (M, g) the Morrey-Sobolev inequality

$$\|u\|_{L^\infty(M)} \leq C \|u\|_{L^1(M)}^{1-\eta} \|\nabla_g u\|_{L^p(M)}^\eta, \quad \forall u \in \text{Lip}_0(M), \quad (\mathbf{MS})_C^2$$

where

$$(1.4) \quad \eta = \frac{np}{np + p - n}.$$

Talenti [19, Theorem 2.C] proved that $(\mathbf{MS})_{\mathbf{C}_2(p,n)}^2$ holds on (\mathbb{R}^n, e) with the sharp constant

$$\mathbf{C}_2(p, n) = (n\omega_n^{\frac{1}{n}})^{-\frac{np'}{n+p'}} \left(\frac{1}{n} + \frac{1}{p'} \right) \left(\frac{1}{n} - \frac{1}{p} \right)^{\frac{(n-1)p'-n}{n+p'}} \left(\mathbf{B} \left(\frac{1-n}{n}p' + 1, p' + 1 \right) \right)^{\frac{n}{n+p'}},$$

where $\mathbf{B}(\cdot, \cdot)$ stands for the Euler beta-function. The unique family of extremals (up to a constant multiplication) is given by

$$v_{\lambda, x_0}(x) = \begin{cases} \int_{|x-x_0|}^{\lambda} r^{\frac{1-n}{p-1}} (\lambda^n - r^n)^{\frac{1}{p-1}} dr, & \text{if } |x - x_0| \leq \lambda; \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$, $x_0 \in \mathbb{R}^n$.

Similar results can be obtained for $(\mathbf{MS})_{\mathbf{C}_2(p,n)}^2$ as in Theorems 1.1 & 1.2; namely, we prove:

Theorem 1.3. *Let (M, g) be an n -dimensional Cartan-Hadamard manifold which verifies the Cartan-Hadamard conjecture in the same dimension, and let $p > n$.*

- (i) [Sharpness] *The Morrey-Sobolev inequality $(\mathbf{MS})_{\mathbf{C}_2(p,n)}^2$ holds on (M, g) ; moreover, $\mathbf{C}_2(p, n)$ is sharp, i.e.,*

$$\mathbf{C}_2(p, n)^{-1} = \inf_{u \in \text{Lip}_0(M) \setminus \{0\}} \frac{\|u\|_{L^1(M)}^{1-\eta} \|\nabla_g u\|_{L^p(M)}^{\eta}}{\|u\|_{L^\infty(M)}},$$

where η is given by (1.4).

- (ii) [Extremals] *Let $x_0 \in M$. The following statements are equivalent:*
- (a) *For every $\kappa > 0$ there exists a non-negative extremal function $u \in \text{Lip}_0(M)$ in $(\mathbf{MS})_{\mathbf{C}_2(p,n)}^2$, concentrated around x_0 and $\mathcal{H}^n(\text{spt } u) = \kappa$;*
 - (b) *(M, g) is isometric to (\mathbb{R}^n, e) .*

Theorem 1.4. *Let (M, g) be a complete, n -dimensional Riemannian manifold with non-negative Ricci curvature, let $p > n$, and assume that $(\mathbf{MS})_C^2$ holds on (M, g) for some $C > 0$. Then the following assertions hold:*

- (i) *$C \geq \mathbf{C}_2(p, n)$;*
- (ii) *(M, g) has the large volume balls property, i.e.,*

$$\text{Vol}_g(B(x, \rho)) \geq \left(\frac{\mathbf{C}_2(p, n)}{C} \right)^{\frac{pn}{p-n} + 1} \omega_n \rho^n \text{ for all } x \in M, \rho \geq 0;$$

- (iii) *$(\mathbf{MS})_{\mathbf{C}_2(p,n)}^2$ holds on (M, g) if and only if (M, g) is isometric to (\mathbb{R}^n, e) .*

Organization of the paper. In Section 2 we recall the notions and results from Riemannian geometry which are used throughout the proofs. In Section 3 we deal with the sharp Morrey-Sobolev inequality with support-bound, providing the proof of Theorems 1.1 and 1.2. In Section 4 we treat the sharp Morrey-Sobolev inequality with L^1 -bound, proving Theorems 1.3 and 1.4.

2. PRELIMINARIES

Let (M, g) be an complete n -dimensional Riemannian manifold, and $d : M \times M \rightarrow [0, \infty)$ be the metric function associated to the Riemannian metric g . Let $B(x, \rho) = \{y \in M : d(x, y) < \rho\}$ be the open geodesic ball with center $x \in M$ and radius $\rho > 0$. If dV_g is the canonical volume element on (M, g) , the volume of an open bounded set $S \subset M$ is $\text{Vol}_g(S) = \int_S dV_g = \mathcal{H}^n(S)$, where $\mathcal{H}^n(S)$ is the n -dimensional Hausdorff measure of S with respect to the metric function d . If $d\sigma_g$ denotes the $(n-1)$ -dimensional Riemann measure induced on ∂S by g , $\text{Area}_g(\partial S) = \int_{\partial S} d\sigma_g = \mathcal{H}^{n-1}(\partial S)$ denotes the area of ∂S with respect to the metric g . In general, one has for every $x \in M$ that

$$(2.1) \quad \lim_{\rho \rightarrow 0^+} \frac{\text{Vol}_g(B(x, \rho))}{\omega_n \rho^n} = 1,$$

where ω_n is the volume of the standard n -dimensional Euclidean unit ball. As usual, $B_e(0, \delta)$, dx , $d\sigma_e$, $\text{Vol}_e(S)$ and $\text{Area}_e(S)$ denote the Euclidean counterparts of the above notions when $S \subset \mathbb{R}^n$.

Let $p > 1$. The norm of $L^p(M)$ is given by $\|u\|_{L^p(M)} = \left(\int_M |u|^p dV_g \right)^{\frac{1}{p}}$. Let $u : M \rightarrow \mathbb{R}$ be a function of class C^1 . If (x^i) denotes the local coordinate system on a coordinate neighborhood of $x \in M$, and the local components of the differential of u are denoted by $u_i = \frac{\partial u}{\partial x_i}$, then the local components of the gradient $\nabla_g u$ are $u^i = g^{ij} u_j$. Here, g^{ij} are the local components of $g^{-1} = (g_{ij})^{-1}$. The $L^p(M)$ norm of $\nabla_g u(x) \in T_x M$ is given by $\|\nabla_g u\|_{L^p(M)} = \left(\int_M |\nabla_g u|^p dV_g \right)^{\frac{1}{p}}$. If $u \in \text{Lip}_0(M)$, i.e., $u : M \rightarrow \mathbb{R}$ is a Lipschitz function with compact support, then it is a.e. differentiable on M and $\|\nabla_g u\|_{L^p(M)}$ is well-defined. The space $W^{1,p}(M)$ is the completion of $C_0^\infty(M)$ w.r.t. the norm $\|u\|_{W^{1,p}(M)} = \|u\|_{L^p(M)} + \|\nabla_g u\|_{L^p(M)}$.

In the proof of our results Bishop-Gromov-type volume comparison principles play a crucial role. On account of Wu and Xin [17, Theorems 6.1 & 6.3], we adapt the following version:

Theorem 2.1. [Volume comparison] *Let (M, g) be a complete, n -dimensional Riemannian manifold and $x_0 \in M$. Then the following statements hold.*

- (a) *If (M, g) is a Cartan-Hadamard manifold, the function $\rho \mapsto \frac{\text{Vol}_g(B(x_0, \rho))}{\rho^n}$ is non-decreasing, $\rho > 0$. In particular, from (2.1) we have*

$$(2.2) \quad \text{Vol}_g(B(x_0, \rho)) \geq \omega_n \rho^n \text{ for all } \rho > 0.$$

If equality holds in (2.2), then the sectional curvature is identically zero.

- (b) *If (M, g) has non-negative Ricci curvature, the function $\rho \mapsto \frac{\text{Vol}_g(B(x_0, \rho))}{\rho^n}$ is non-increasing, $\rho > 0$. In particular, from (2.1) we have*

$$(2.3) \quad \text{Vol}_g(B(x_0, \rho)) \leq \omega_n \rho^n \text{ for all } \rho > 0.$$

If equality holds in (2.3), then the sectional curvature is identically zero.

3. SHARP MORREY-SOBOLEV INEQUALITY WITH SUPPORT-BOUND

Let (M, g) be a complete n -dimensional Riemannian manifold, and let $p > n$. For $C > 0$, we recall the Morrey-Sobolev inequality $(\mathbf{MS})_C^1$ with support-bound, i.e.,

$$\|u\|_{L^\infty(M)} \leq C \mathcal{H}^n(\text{sprt } u)^{\frac{1}{n} - \frac{1}{p}} \|\nabla_g u\|_{L^p(M)}, \quad \forall u \in \text{Lip}_0(M).$$

We first present a result inspired by Aubin [1] and Hebey [12].

Proposition 3.1. *If $(\mathbf{MS})_C^1$ holds, then $C \geq C_1(p, n)$.*

Proof. Assume by contradiction that $C < C_1(p, n)$. Let $x_0 \in M$. For every $\varepsilon > 0$, there exists a local chart (Ω, ϕ) of M at the point x_0 and a number $\delta > 0$ such that $\phi(\Omega) = B_e(0, \delta)$ and the components g_{ij} of the metric g satisfy

$$(3.1) \quad (1 - \varepsilon)\delta_{ij} \leq g_{ij} \leq (1 + \varepsilon)\delta_{ij}$$

in the sense of bilinear forms.

Due to $(\mathbf{MS})_C^1$ and to the two-sided estimate (3.1), for $\varepsilon > 0$ small enough, there exists $\tilde{\delta} > 0$ and $C' < C_1(p, n)$ such that for every $\delta \in (0, \tilde{\delta})$ and $w \in \text{Lip}_0(B_e(0, \delta))$,

$$(3.2) \quad \|w\|_{L^\infty(B_e(0, \delta))} \leq C' \mathcal{H}^n(\text{sprt } w)^{\frac{1}{n} - \frac{1}{p}} \|\nabla w\|_{L^p(B_e(0, \delta))}.$$

Let $u \in \text{Lip}_0(\mathbb{R}^n)$ be arbitrarily fixed and set $w_\lambda(x) = u(\lambda x)$, $\lambda > 0$. For enough large $\lambda > 0$, one has $w_\lambda \in \text{Lip}_0(B_e(0, \delta))$. Replacing w_λ into (3.2), and using the scaling properties

$$\|w_\lambda\|_{L^\infty(B_e(0, \delta))} = \|u\|_{L^\infty(\mathbb{R}^n)}, \quad \mathcal{H}^n(\text{sprt } w_\lambda) = \lambda^{-n} \mathcal{H}^n(\text{sprt } u),$$

and

$$\int_{B_e(0, \delta)} |\nabla w_\lambda|^p dx = \lambda^{p-n} \int_{\mathbb{R}^n} |\nabla u|^p dx,$$

one has

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C' \mathcal{H}^n(\text{sprt } u)^{\frac{1}{n} - \frac{1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Inserting the function introduced in (1.3) into the latter relation, we obtain $C_1(p, n) \leq C'$, a contradiction. \square

Proof of Theorem 1.1. The first part of the proof is similar to Druet, Hebey and Vaugon [9], see also Aubin, Druet and Hebey [2]; since some intermediate steps will be crucial in the second part (i.e., in the existence of extremal functions), we shall present its complete proof. Let $p > n$.

(i) Clearly, it is enough to consider only non-negative test functions in the Morrey-Sobolev inequality $(\mathbf{MS})_C^1$. Moreover, by standard approximation/density argument and Morse theory, it is sufficient to deal with continuous test functions $u : M \rightarrow [0, \infty)$ having compact support $S \subset M$, where S is an enough smooth set, u being of class C^2 in S and having only non-degenerate critical points in S . Fixing such a function $u : M \rightarrow [0, \infty)$, we associate to u its Euclidean decreasing rearrangement function $u^* : \mathbb{R}^n \rightarrow [0, \infty)$ which is radially symmetric and is defined for every $t > 0$ by

$$(3.3) \quad \text{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\}) = \text{Vol}_g(\{x \in M : u(x) > t\}) \stackrel{\text{def.}}{=} V(t).$$

By definition, u^* is a Lipschitz function with compact support, and

$$(3.4) \quad \|u\|_{L^\infty(M)} = \|u^*\|_{L^\infty(\mathbb{R}^n)}, \quad \text{Vol}_g(\text{sprt } u) = \text{Vol}_e(\text{sprt } u^*).$$

On one hand, for every $0 < t < \|u\|_{L^\infty(M)}$, we consider the level sets

$$\Gamma_t = u^{-1}(t) \subset S \subset M, \quad \Gamma_t^* = (u^*)^{-1}(t) \subset \mathbb{R}^n.$$

Since u^* is radial, Γ_t^* is an $(n-1)$ -dimensional sphere with $\Gamma_t^* = \partial(\{x \in \mathbb{R}^n : u^*(x) > t\})$ for every $0 < t < \|u\|_{L^\infty(M)}$, and

$$\text{Area}_e(\Gamma_t^*) = n\omega_n^{\frac{1}{n}} \text{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\})^{\frac{n-1}{n}}.$$

In particular, the latter relation, the validity of Cartan-Hadamard conjecture and (3.3) imply that

$$(3.5) \quad \text{Area}_g(\Gamma_t) \geq \text{Area}_e(\Gamma_t^*) \quad \text{for every } 0 < t < \|u\|_{L^\infty(M)}.$$

A simple application of the co-area formula (see Chavel [4, p. 302]) and (3.3) give

$$(3.6) \quad V'(t) = - \int_{\Gamma_t} \frac{1}{|\nabla_g u|} d\sigma_g = - \int_{\Gamma_t^*} \frac{1}{|\nabla u^*|} d\sigma_e.$$

Since $|\nabla u^*|$ is constant on the sphere Γ_t^* , the second relation from (3.6) gives that

$$(3.7) \quad V'(t) = - \frac{\text{Area}_e(\Gamma_t^*)}{|\nabla u^*(x)|}, \quad x \in \Gamma_t^*.$$

On the other hand, by Hölder's inequality and the first relation of (3.6), one has

$$\text{Area}_g(\Gamma_t) = \int_{\Gamma_t} d\sigma_g \leq (-V'(t))^{\frac{p-1}{p}} \left(\int_{\Gamma_t} |\nabla_g u|^{p-1} d\sigma_g \right)^{\frac{1}{p}}.$$

Consequently, by (3.5) and (3.7), for every $0 < t < \|u\|_{L^\infty(M)}$ we have

$$\begin{aligned} \int_{\Gamma_t} |\nabla_g u|^{p-1} d\sigma_g &\geq \text{Area}_g(\Gamma_t)^p (-V'(t))^{1-p} \\ &\geq \text{Area}_e(\Gamma_t^*)^p \left(\frac{\text{Area}_e(\Gamma_t^*)}{|\nabla u^*(x)|} \right)^{1-p} \quad (x \in \Gamma_t^*) \\ &= \int_{\Gamma_t^*} |\nabla u^*|^{p-1} d\sigma_e. \end{aligned}$$

By the co-area formula and the latter inequality, an integration with respect to t gives

$$(3.8) \quad \int_M |\nabla_g u|^p dV_g \geq \int_{\mathbb{R}^n} |\nabla u^*|^p dx.$$

Applying Talenti's inequality for the function $u^* : \mathbb{R}^n \rightarrow \mathbb{R}$ (see [19, Theorem 2.E] and Introduction), relations (3.4) and (3.8) provide

$$\begin{aligned} \|u\|_{L^\infty(M)} &= \|u^*\|_{L^\infty(\mathbb{R}^n)} \\ &\leq C_1(p, n) \mathcal{H}^n(\text{sprt } u^*)^{\frac{1}{n} - \frac{1}{p}} \|\nabla u^*\|_{L^p(\mathbb{R}^n)} \\ (3.9) \quad &\leq C_1(p, n) \mathcal{H}^n(\text{sprt } u)^{\frac{1}{n} - \frac{1}{p}} \|\nabla_g u\|_{L^p(M)}, \end{aligned}$$

which is precisely $(\mathbf{MS})_{C_1(p,n)}^1$ on (M, g) . Moreover, this inequality and Proposition 3.1 show that

$$C_1(p, n)^{-1} = \inf_{u \in \text{Lip}_0(M) \setminus \{0\}} \frac{\mathcal{H}^n(\text{sprt } u)^{\frac{1}{n} - \frac{1}{p}} \|\nabla_g u\|_{L^p(M)}}{\|u\|_{L^\infty(M)}}.$$

(ii) By Talenti's result, we clearly have (b) \Rightarrow (a). Let $x_0 \in M$, and assume that (a) holds, i.e., for every $\kappa > 0$ there exists a non-negative extremal function $u \in \text{Lip}_0(M)$ in $(\mathbf{MS})_{C_1(p,n)}^1$, concentrated around x_0 and $\mathcal{H}^n(\text{sprt } u) = \kappa$. Therefore, in (3.9) we have equalities; in particular, the Euclidean decreasing rearrangement function $u^* : \mathbb{R}^n \rightarrow [0, \infty)$ associated to u verifies

$$\|u^*\|_{L^\infty(\mathbb{R}^n)} = C_1(p, n) \mathcal{H}^n(\text{sprt } u^*)^{\frac{1}{n} - \frac{1}{p}} \|\nabla u^*\|_{L^p(\mathbb{R}^n)}.$$

According to Talenti's result, since u^* is an extremal in $(\mathbf{MS})_{C_1(p,n)}^1$ on (\mathbb{R}^n, e) , it has the shape from (1.3), i.e.,

$$u^*(x) = \left(\lambda^{\frac{p-n}{p-1}} - |x|^{\frac{p-n}{p-1}} \right)_+, \quad x \in \mathbb{R}^n.$$

In addition, since $\mathcal{H}^n(\text{sprt } u^*) = \mathcal{H}^n(\text{sprt } u) = \kappa$, we have $\lambda = (\kappa \omega_n^{-1})^{\frac{1}{n}}$. Consequently, for every $0 < t < \|u\|_{L^\infty(M)} = \|u^*\|_{L^\infty(\mathbb{R}^n)} = \lambda^{\frac{p-n}{p-1}}$, one has $\{x \in \mathbb{R}^n : u^*(x) > t\} = B_e(0, \rho_t)$, where

$$\rho_t = \left(\lambda^{\frac{p-n}{p-1}} - t \right)^{\frac{p-1}{p-n}}.$$

Fix $0 < t < \lambda^{\frac{p-n}{p-1}}$. We claim that

$$\{x \in M : u(x) > t\} = B(x_0, \rho_t).$$

On one hand, since u is concentrated around x_0 , there exists $\rho'_t > 0$ such that $\{x \in M : u(x) > t\} = B(x_0, \rho'_t)$. Therefore, the claim is concluded once we prove that $\rho'_t = \rho_t$. Due to (3.3), one has

$$(3.10) \quad \text{Vol}_g(B(x_0, \rho'_t)) = \text{Vol}_e(B_e(0, \rho_t)).$$

On the other hand, since u is an extremal in $(\mathbf{MS})_{C_1(p,n)}^1$, we have equalities not only in (3.9) but also in (3.8). Subsequently, we have equality also in (3.5), i.e.,

$$\text{Area}_g(\Gamma_t) = \text{Area}_e(\Gamma_t^*).$$

This relation together with (3.3) imply that we have equality case in the Cartan-Hadamard conjecture, i.e., $\{x \in M : u(x) > t\} = B(x_0, \rho'_t)$ is isometric to the n -dimensional Euclidean ball with volume $\text{Vol}_g(B(x_0, \rho'_t))$. On account of (3.10), we actually have that $B(x_0, \rho'_t)$ and $B_e(0, \rho_t)$ are isometric, thus $\rho'_t = \rho_t$. Therefore, $\text{Vol}_g(B(x_0, \rho_t)) = \omega_n \rho_t^n$. If $t \rightarrow 0^+$, the latter relation implies that $\text{Vol}_g(B(x_0, \lambda)) = \omega_n \lambda^n$. Due to the arbitrariness of $\kappa > 0$, so $\lambda = (\kappa \omega_n^{-1})^{\frac{1}{n}}$, we have that

$$\text{Vol}_g(B(x_0, \rho)) = \omega_n \rho^n \quad \text{for all } \rho > 0.$$

By Theorem 2.1 (i) we have that the sectional curvature on the Cartan-Hadamard manifold (M, g) is identically zero, which concludes the proof. \square

Proof of Theorem 1.2. We assume that $(\mathbf{MS})_C^1$ holds on (M, g) for some $C > 0$. By Proposition 3.1, we already have that $C \geq C_1(p, n)$, i.e., (i) is proved.

(ii) We are going to prove that (M, g) has the large volume balls property, i.e.,

$$\text{Vol}_g(B(x, \rho)) \geq \left(\frac{C_1(p, n)}{C} \right)^{\frac{pn}{p-n}} \omega_n \rho^n \text{ for all } x \in M, \rho \geq 0.$$

Let $x_0 \in M$ be fixed. For every $\lambda > 0$, we consider the function

$$u_\lambda(x) = \left(\lambda^{\frac{p-n}{p-1}} - d(x_0, x)^{\frac{p-n}{p-1}} \right)_+, \quad x \in M.$$

It is clear that $u_\lambda \in \text{Lip}_0(M)$ and

$$(3.11) \quad \|u_\lambda\|_{L^\infty(M)} = \lambda^{\frac{p-n}{p-1}}, \quad \mathcal{H}^n(\text{spt } u_\lambda) = \text{Vol}_g(B(x_0, \lambda)).$$

The chain rule (see Hebey [12, Proposition 2.5]) implies that

$$\nabla_g u_\lambda(x) = -\frac{p-n}{p-1} d(x_0, x)^{\frac{1-n}{p-1}} \nabla_g d(x_0, x), \quad x \in B(x_0, \lambda).$$

Taking into account that $|\nabla_g d(x_0, x)| = 1$ for a.e. $x \in M$, the layer cake representation and Theorem 2.1 (ii) give that

$$\begin{aligned} \int_M |\nabla_g u_\lambda|^p dV_g &= \int_{B(x_0, \lambda)} |\nabla_g u_\lambda|^p dV_g \\ &= \left(\frac{p-n}{p-1} \right)^p \int_{B(x_0, \lambda)} d(x_0, x)^{\frac{p(1-n)}{p-1}} dV_g \\ &= \left(\frac{p-n}{p-1} \right)^p \int_0^\infty \text{Vol}_g \left(\{x \in B(x_0, \lambda) : d(x_0, x)^{\frac{p(1-n)}{p-1}} > t\} \right) dt \\ &= \left(\frac{p-n}{p-1} \right)^p \int_{\lambda^{\frac{p(1-n)}{p-1}}}^\infty \text{Vol}_g \left(\{x \in B(x_0, \lambda) : d(x_0, x)^{\frac{p(1-n)}{p-1}} > t\} \right) dt \\ &\quad + \left(\frac{p-n}{p-1} \right)^p \int_0^{\lambda^{\frac{p(1-n)}{p-1}}} \text{Vol}_g \left(\{x \in B(x_0, \lambda) : d(x_0, x)^{\frac{p(1-n)}{p-1}} > t\} \right) dt \\ &= \left(\frac{p-n}{p-1} \right)^p \frac{p(n-1)}{p-1} \int_0^\lambda \text{Vol}_g(B(x_0, \rho)) \rho^{\frac{-pn+1}{p-1}} d\rho \\ &\quad + \left(\frac{p-n}{p-1} \right)^p \text{Vol}_g(B(x_0, \lambda)) \lambda^{\frac{p(1-n)}{p-1}} \\ &\leq \left(\frac{p-n}{p-1} \right)^p \omega_n \left[\frac{p(n-1)}{p-1} \int_0^\lambda \rho^{n+\frac{-pn+1}{p-1}} d\rho + \lambda^{n+\frac{p(1-n)}{p-1}} \right] \\ &= \left(\frac{p-n}{p-1} \right)^{p-1} n \omega_n \lambda^{\frac{p-n}{p-1}}. \end{aligned}$$

Inserting u_λ into $(\mathbf{MS})_C^1$, relation (3.11) and the above estimate yield that

$$\lambda^{\frac{p-n}{p-1}} \leq C \text{Vol}_g(B(x_0, \lambda))^{\frac{1}{n}-\frac{1}{p}} \left(\frac{p-n}{p-1} \right)^{\frac{1}{p'}} (n \omega_n)^{\frac{1}{p}} \lambda^{\frac{p-n}{p(p-1)}}.$$

Reorganizing this inequality and taking into account the form of the constant $C_1(p, n)$, see (1.2), it turns out that for every $\lambda > 0$, we have

$$(3.12) \quad \text{Vol}_g(B(x_0, \lambda)) \geq \left(\frac{C_1(p, n)}{C} \right)^{\frac{pn}{p-n}} \omega_n \lambda^n.$$

Let $x \in M$ and $\rho > 0$ be fixed arbitrarily. Then, one has

$$\begin{aligned} \frac{\text{Vol}_g(B(x, \rho))}{\omega_n \rho^n} &\geq \limsup_{r \rightarrow \infty} \frac{\text{Vol}_g(B(x, r))}{\omega_n r^n} && [\text{cf. Theorem 2.1 (ii)}] \\ &\geq \limsup_{r \rightarrow \infty} \frac{\text{Vol}_g(B(x_0, r - d(x_0, x)))}{\omega_n r^n} && [B(x, r) \supset B(x_0, r - d(x_0, x))] \\ &= \limsup_{r \rightarrow \infty} \left(\frac{\text{Vol}_g(B(x_0, r - d(x_0, x)))}{\omega_n (r - d(x_0, x))^n} \cdot \frac{(r - d(x_0, x))^n}{r^n} \right) \\ &\geq \left(\frac{C_1(p, n)}{C} \right)^{\frac{pn}{p-n}}, && [\text{cf. (3.12)}] \end{aligned}$$

which concludes the proof of (ii).

(iii) By Talenti's result, if (M, g) is isometric to (\mathbb{R}^n, e) , then $(\mathbf{MS})_{C_1(p, n)}^1$ holds. Conversely, let us assume that $(\mathbf{MS})_{C_1(p, n)}^1$ holds on (M, g) . First, by (ii) we have that $\text{Vol}_g(B(x, \rho)) \geq \omega_n \rho^n$ for every $x \in M$ and $\rho > 0$. By (2.3), we also have the converse inequality $\text{Vol}_g(B(x, \rho)) \leq \omega_n \rho^n$, thus

$$(3.13) \quad \text{Vol}_g(B(x, \rho)) = \omega_n \rho^n \quad \text{for all } x \in M, \rho > 0.$$

By Theorem 2.1 (ii), it follows that the sectional curvature on (M, g) is identically zero. Then relation (3.13) implies that (M, g) is isometric to (\mathbb{R}^n, e) . \square

4. SHARP MORREY-SOBOLEV INEQUALITY WITH L^1 -BOUND

The structure of this section is similar to the previous one; in the sequel, we shall point out the differences. As before, let (M, g) be a complete n -dimensional Riemannian manifold, and let $p > n$. For $C > 0$, we recall the Morrey-Sobolev inequality with L^1 -bound, i.e.,

$$\|u\|_{L^\infty(M)} \leq C \|u\|_{L^1(M)}^{1-\eta} \|\nabla_g u\|_{L^p(M)}^\eta, \quad \forall u \in \text{Lip}_0(M), \quad (\mathbf{MS})_C^2$$

where

$$\eta = \frac{np}{np + p - n}.$$

Proposition 4.1. *If $(\mathbf{MS})_C^2$ holds, then $C \geq C_2(p, n)$.*

Proof. We follow the proof of Proposition 3.1. The only minor difference is that we use a further scaling property. Namely, let $u \in \text{Lip}_0(\mathbb{R}^n)$ and $w_\lambda(x) = u(\lambda x)$, $\lambda > 0$. Then for enough large $\lambda > 0$, one has $w_\lambda \in \text{Lip}_0(B_e(0, \delta))$, and in addition to the scaling properties from Proposition 3.1 we also have $\|w_\lambda\|_{L^1(\mathbb{R}^n)} = \lambda^{-n} \|u\|_{L^1(\mathbb{R}^n)}$. \square

Proof of Theorem 1.3. Let $p > n$. Let $u : M \rightarrow [0, \infty)$ be a function with the same properties as in the proof of Theorem 1.1. If we associate to u its Euclidean decreasing rearrangement function $u^* : \mathbb{R}^n \rightarrow [0, \infty)$ which is radially symmetric and defined by (3.3), one has that

$$\|u\|_{L^\infty(M)} = \|u^*\|_{L^\infty(\mathbb{R}^n)}, \quad \|\nabla_g u\|_{L^p(M)} \geq \|\nabla u^*\|_{L^p(\mathbb{R}^n)},$$

see (3.4) and (3.8), respectively. In addition, by the layer cake representation and (3.3), we also have

$$\begin{aligned} \|u\|_{L^1(M)} &= \int_0^\infty \text{Vol}_g(\{x \in M : u(x) > t\}) dt = \int_0^\infty \text{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\}) dt \\ &= \|u^*\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Consequently, the latter relations and Talenti's result (see [19, Theorem 2.C] and the Introduction) imply that

$$\begin{aligned} \|u\|_{L^\infty(M)} &= \|u^*\|_{L^\infty(\mathbb{R}^n)} \\ &\leq C_2(p, n) \|u^*\|_{L^1(\mathbb{R}^n)}^{1-\eta} \|\nabla u^*\|_{L^p(\mathbb{R}^n)}^\eta \\ (4.1) \quad &\leq C_2(p, n) \|u\|_{L^1(M)}^{1-\eta} \|\nabla_g u\|_{L^p(M)}^\eta, \end{aligned}$$

i.e., $(\mathbf{MS})_{C_2(p, n)}^2$ holds on (M, g) . The sharpness of the constant $C_2(p, n)$ follows by the latter estimate and Proposition 4.1, concluding the proof of (i).

Before to provide the proof of (ii), we introduce some notations which will be useful in the sequel. Let $\lambda > 0$ and define $f_\lambda : (0, \lambda] \rightarrow [0, \infty)$ and $F_\lambda : [0, \lambda] \rightarrow [0, \infty)$ by

$$(4.2) \quad f_\lambda(r) = r^{\frac{1-n}{p-1}} (\lambda^n - r^n)^{\frac{1}{p-1}}, \quad F_\lambda(s) = \int_0^s f_\lambda(r) dr.$$

(ii) Let $x_0 \in M$ and $\kappa > 0$ be fixed arbitrarily and let $u \in \text{Lip}_0(M)$ be a non-negative extremal function in $(\mathbf{MS})_{C_2(p, n)}^2$, concentrated around x_0 and $\mathcal{H}^n(\text{sprt } u) = \kappa$. Since we have equalities in (4.1), the function $u^* : \mathbb{R}^n \rightarrow [0, \infty)$ verifies

$$\|u^*\|_{L^\infty(\mathbb{R}^n)} = C_2(p, n) \|u^*\|_{L^1(\mathbb{R}^n)}^{1-\eta} \|\nabla u^*\|_{L^p(\mathbb{R}^n)}^\eta.$$

Since u^* is an extremal in $(\mathbf{MS})_{C_2(p, n)}^2$ on (\mathbb{R}^n, e) , by Talenti's result, its expression is given by

$$u^*(x) = \begin{cases} F_\lambda(\lambda) - F_\lambda(|x|), & \text{if } x \in B_e(0, \lambda); \\ 0, & \text{if } x \notin B_e(0, \lambda). \end{cases}$$

Note that by $\mathcal{H}^n(\text{sprt } u^*) = \mathcal{H}^n(\text{sprt } u) = \kappa$, we have $\lambda = (\kappa \omega_n^{-1})^{\frac{1}{n}}$. Moreover,

$$\|u\|_{L^\infty(M)} = \|u^*\|_{L^\infty(\mathbb{R}^n)} = F_\lambda(\lambda) = \frac{\lambda^{p'}}{n} \mathbf{B}\left(\frac{1-n}{n} p' + 1, p'\right)$$

and since F_λ is increasing on $[0, \lambda]$, for every $0 < t < F_\lambda(\lambda)$ one has $\{x \in \mathbb{R}^n : u^*(x) > t\} = B_e(0, \rho_t)$, where

$$\rho_t = F_\lambda^{-1}(F_\lambda(\lambda) - t).$$

A similar argument as in the proof of Theorem 1.1 (ii) shows that for every $0 < t < F_\lambda(\lambda)$, we have $\{x \in M : u(x) > t\} = B(x_0, \rho_t)$, and finally

$$\text{Vol}_g(B(x_0, \rho)) = \omega_n \rho^n \quad \text{for all } \rho > 0,$$

which concludes the proof. \square

Proof of Theorem 1.4. (i) By Proposition 4.1, we have that $C \geq C_2(p, n)$ whenever $(\mathbf{MS})_C^2$ is assumed to hold on (M, g) .

(ii) Let $x_0 \in M$ be fixed. By using (4.2), for every $\lambda > 0$ we consider the functions $u_\lambda \in \text{Lip}_0(M)$ and $w_\lambda \in \text{Lip}_0(\mathbb{R}^n)$ defined by

$$u_\lambda(x) = \begin{cases} F_\lambda(\lambda) - F_\lambda(d(x_0, x)), & \text{if } x \in B(x_0, \lambda); \\ 0, & \text{if } x \notin B(x_0, \lambda), \end{cases}$$

and

$$w_\lambda(x) = \begin{cases} F_\lambda(\lambda) - F_\lambda(|x|), & \text{if } x \in B_e(0, \lambda); \\ 0, & \text{if } x \notin B_e(0, \lambda). \end{cases}$$

Since u_λ verifies $(\mathbf{MS})_C^2$ on (M, g) , and w_λ is an extremal in $(\mathbf{MS})_{C_2(p, n)}^2$ on (\mathbb{R}^n, e) , we have that

$$(4.3) \quad \|u_\lambda\|_{L^\infty(M)} \leq C \|u_\lambda\|_{L^1(M)}^{1-\eta} \|\nabla_g u_\lambda\|_{L^p(M)}^\eta$$

and

$$(4.4) \quad \|w_\lambda\|_{L^\infty(\mathbb{R}^n)} = C_2(p, n) \|w_\lambda\|_{L^1(\mathbb{R}^n)}^{1-\eta} \|\nabla w_\lambda\|_{L^p(\mathbb{R}^n)}^\eta.$$

Moreover, by the above definitions and a simple computation give

$$(4.5) \quad \|u_\lambda\|_{L^\infty(M)} = \|w_\lambda\|_{L^\infty(\mathbb{R}^n)} = F_\lambda(\lambda) = \frac{\lambda^{p'}}{n} \mathbf{B}\left(\frac{1-n}{n}p' + 1, p'\right).$$

Similar computations show that

$$(4.6) \quad \|w_\lambda\|_{L^1(\mathbb{R}^n)} = \omega_n \int_0^\lambda \rho^n f_\lambda(\rho) d\rho = \frac{\lambda^{n+p'} \omega_n}{n} \mathbf{B}\left(\frac{1-n}{n}p' + 2, p'\right)$$

and

$$\|\nabla w_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{\frac{n+p'}{p}} \omega_n^{\frac{1}{p}} \mathbf{B}\left(\frac{1-n}{n}p' + 1, p' + 1\right)^{\frac{1}{p}}.$$

In the sequel, we shall estimate $\|u_\lambda\|_{L^1(M)}$ and $\|\nabla_g u_\lambda\|_{L^p(M)}$. First, by the layer cake representation, one has

$$\begin{aligned} \|u_\lambda\|_{L^1(M)} &= \int_M u_\lambda(x) dV_g = \int_{B(x_0, \lambda)} (F_\lambda(\lambda) - F_\lambda(d(x_0, x))) dV_g \\ &= \int_0^\infty \text{Vol}_g(\{x \in B(x_0, \lambda) : F_\lambda(\lambda) - F_\lambda(d(x_0, x)) > t\}) dt \\ &\quad \text{[change of var. } t = F_\lambda(\lambda) - F_\lambda(\rho)] \\ &= \int_0^\lambda \text{Vol}_g(B(x_0, \rho)) f_\lambda(\rho) d\rho. \end{aligned}$$

Then, since $\nabla_g u_\lambda(x) = -f_\lambda(d(x_0, x))\nabla_g d(x_0, x)$ for every $x \in B(x_0, \lambda)$, by Bishop-Gromov comparison theorem (see relation (2.3)), one has

$$\begin{aligned}
\|\nabla_g u_\lambda\|_{L^p(M)}^p &= \int_{B(x_0, \lambda)} f_\lambda(d(x_0, x))^p dV_g \\
&= \int_{B(x_0, \lambda)} (\lambda^n d(x_0, x)^{1-n} - d(x_0, x))^{p'} dV_g \\
&= \int_0^\infty \text{Vol}_g(\{x \in B(x_0, \lambda) : (\lambda^n d(x_0, x)^{1-n} - d(x_0, x))^{p'} > t\}) dt \\
&\quad \left[\text{change of var. } t = (\lambda^n \rho^{1-n} - \rho)^{p'} \right] \\
&= p' \int_0^\lambda \text{Vol}_g(B(x_0, \rho)) (\lambda^n \rho^{1-n} - \rho)^{p'-1} ((n-1)\lambda^n \rho^{-n} + 1) d\rho \\
&\leq p' \int_0^\lambda (\lambda^n \rho^{1-n} - \rho)^{p'-1} ((n-1)\lambda^n + \rho^n) d\rho \\
&= \lambda^{n+p'} \omega_n \mathbf{B}\left(\frac{1-n}{n} p' + 1, p' + 1\right) \\
&= \|\nabla w_\lambda\|_{L^p(\mathbb{R}^n)}^p.
\end{aligned}$$

Subtracting (4.4) from (4.3), relations (4.5), (4.6) and the above computations give that for every $\lambda > 0$,

$$(4.7) \quad \int_0^\lambda \left(\text{Vol}_g(B(x_0, \rho)) - \left(\frac{\mathbf{C}_2(p, n)}{C} \right)^{\frac{1}{1-\eta}} \omega_n \rho^n \right) f_\lambda(\rho) d\rho \geq 0.$$

We claim that

$$(4.8) \quad \ell_\infty^{x_0} := \limsup_{\rho \rightarrow \infty} \frac{\text{Vol}_g(B(x_0, \rho))}{\omega_n \rho^n} \geq \left(\frac{\mathbf{C}_2(p, n)}{C} \right)^{\frac{1}{1-\eta}}.$$

Assuming the contrary, there exists $\varepsilon_0 > 0$ such that for some $\rho_0 > 0$,

$$\frac{\text{Vol}_g(B(x_0, \rho))}{\omega_n \rho^n} \leq \left(\frac{\mathbf{C}_2(p, n)}{C} \right)^{\frac{1}{1-\eta}} - \varepsilon_0, \quad \forall \rho \geq \rho_0.$$

By the latter inequality and (4.7), for every $\lambda > \rho_0$ we obtain that

$$\begin{aligned}
0 &\leq \int_0^\lambda \left(\text{Vol}_g(B(x_0, \rho)) - \left(\frac{\mathbf{C}_2(p, n)}{C} \right)^{\frac{1}{1-\eta}} \omega_n \rho^n \right) f_\lambda(\rho) d\rho \\
&\leq \int_0^{\rho_0} \text{Vol}_g(B(x_0, \rho)) f_\lambda(\rho) d\rho - \varepsilon_0 \omega_n \int_{\rho_0}^\lambda \rho^n f_\lambda(\rho) d\rho \\
&\quad - \left(\frac{\mathbf{C}_2(p, n)}{C} \right)^{\frac{1}{1-\eta}} \omega_n \int_0^{\rho_0} \rho^n f_\lambda(\rho) d\rho.
\end{aligned}$$

Rearranging the above inequality, by (2.3) it follows that

$$(4.9) \quad \varepsilon_0 \int_0^\lambda \rho^n f_\lambda(\rho) d\rho \leq \left(1 - \left(\frac{C_2(p, n)}{C} \right)^{\frac{1}{1-\eta}} + \varepsilon_0 \right) \int_0^{\rho_0} \rho^n f_\lambda(\rho) d\rho.$$

According to (4.6) and to the fact that

$$\int_0^{\rho_0} \rho^n f_\lambda(\rho) d\rho \leq \rho_0^{n+\frac{1-n}{p-1}} \lambda^{\frac{n}{p-1}},$$

inequality (4.9) implies

$$\varepsilon_0 \frac{\lambda^{n+p'}}{n} B\left(\frac{1-n}{n} p' + 2, p'\right) \leq \left(1 - \left(\frac{C_2(p, n)}{C} \right)^{\frac{1}{1-\eta}} + \varepsilon_0 \right) \rho_0^{n+\frac{1-n}{p-1}} \lambda^{\frac{n}{p-1}}, \quad \lambda > \rho_0.$$

Since $n + p' > \frac{n}{p-1}$, letting $\lambda \rightarrow +\infty$ in the latter inequality, we arrive to a contradiction. The proof of (4.8) is complete.

By Theorem 2.1 (ii) and (4.8), it follows that for every $\rho > 0$,

$$\frac{\text{Vol}_g(B(x_0, \rho))}{\omega_n \rho^n} \geq \ell_\infty^{x_0} \geq \left(\frac{C_2(p, n)}{C} \right)^{\frac{1}{1-\eta}}.$$

If $x \in M$ and $\rho > 0$ are arbitrarily fixed, a similar argument applies as in the last step of the proof of Theorem 1.2, where the latter inequality takes the role of (3.12).

(iii) Similar to the proof of Theorem 1.2 (iii). \square

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